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Equations Generated by Means of Parafield Operators

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Abstract

It is shown that using realisations of Lie algebras with parafield operators one can generate infinitely many classes of invariant equations corresponding to each order of parastatistics p.

1. *Introduction*

Recently, many attempts were made to study the properties of different new finite and infinite component relativistically invariant equations. At first sight there was no connection among these quite different types of equations with different properties. However, a method has been proposed by Palev (1969) which allows some classes of these finite and infinite component equations to be considered as different realisations of one and the same generalised equation. It was proved that starting from a given relativistically invariant equation one can generate a class of invariant equations using Bose creation and annihilation operators.

In the present paper we want to show that this procedure of generating new equations from a given one can be considerably extended if, instead of Bose creation and annihilation operators, para-Bose and para-Fermi operators are used. In such a way, infinitely many new classes of equations (except the class generated by the use of Bose operators) are obtained—corresponding to each order of parastatistics of the para-Bose and para-Fermi operators.

The notations used in the paper are introduced in Section 2. In Section 3 a generalisation of the method given by Palev (1969) is described. In Section 4 we discuss the space in which the generated equations are realised.

2. *Some Properties of the Parafield Operators and the Invariant Equations*

We will discuss briefly some elementary notions and properties of the parafield operators and the invariant equations which will be needed further.

We consider the Green algebra† $\mathscr{U}(n,\epsilon)$ in which the following relations for the entities a_i, $\overset{+}{a}_j$, $i,j=1,\ldots,n$ hold‡

$$
\begin{aligned}
&[\tfrac{1}{2}[\overset{+}{a}_i, a_j]_\epsilon, \overset{+}{a}_k]_- = \delta_{jk}\overset{+}{a}_i \\
&[\tfrac{1}{2}[a_i, a_j]_\epsilon, a_k]_- = 0
\end{aligned}
\tag{2.1}
$$

where $\epsilon = \pm$.

All the other relations are obtained by conjugation and Jacobi identity.

It is known, due to Green, that the relations (2.1) can be satisfied by the operators a_i, $\overset{+}{a}_j$ represented in the form

$$
\begin{aligned}
a_i &= \sum_{\alpha=1}^{p} a_i^\alpha \\
\overset{+}{a}_i &= \sum_{\alpha=1}^{p} \overset{+}{a}_i^\alpha
\end{aligned}
\tag{2.2}
$$

and only by them. The index p, which takes arbitrary positive integer values, is called parastatistics of the Green algebra. The entities a_i^α, $\overset{+}{a}_j^\beta$, $i,j=1,\ldots,n$, $\alpha,\beta=1,\ldots,p$ form the quasifield algebra§ $\mathscr{U}(n,p,\epsilon)$ in which the following relations are defined:

$$
\begin{aligned}
&[a_i^\alpha, \overset{+}{a}_j^\alpha]_{-\epsilon} = \delta_{ij}, \quad [a_i^\alpha, a_j^\alpha]_{-\epsilon} = [\overset{+}{a}_i^\alpha, \overset{+}{a}_j^\alpha]_{-\epsilon} = 0 \\
&[a_i^\alpha, \overset{+}{a}_j^\beta]_\epsilon = [a_i^\alpha, a_j^\beta]_\epsilon = [\overset{+}{a}_i^\alpha, \overset{+}{a}_j^\beta]_\epsilon = 0 \qquad \text{if } \alpha \neq \beta
\end{aligned}
\tag{2.3}
$$

The Green Ansatz (2.2) shows that the Green algebra $\mathscr{U}(n,\epsilon)$ is isomorphically embedded in the quasifield algebra $\mathscr{U}(n,p,\epsilon)$.

Let us now remember that the relativistic invariance of the equation

$$(\Gamma^\mu p_\mu - m)\psi(p) = 0 \tag{2.4}$$

means (i) that the wave function $\psi(p)$ belongs to a space X which spans a representation of the Lorentz group $SL(2,C)$, (ii) Γ^μ and m are operators defined in this space through the relations

$$
\begin{aligned}
&[S^{\mu\nu}, \Gamma^\lambda] = i(g^{\nu\lambda}\Gamma^\mu - g^{\mu\lambda}\Gamma^\nu) \\
&[S^{\mu\nu}, m] = 0
\end{aligned}
\tag{2.5}
$$

where $S^{\mu\nu}$ are the generators of the representation of $SL(2,C)$ in X and $g^{00} = -g^{kk} = 1$, that is, Γ^μ is transformed as a four-vector and m as a scalar.

† A precise mathematical definition is given by Kademova (1969).

‡ The operators satisfying the relations (2.1) have been introduced by Green (1953). We call them parafield or Green operators. For $\epsilon = +$ they are known as para-Bose operators, and for $\epsilon = -$ as para-Fermi operators.

§ See Kademova & Palev (1970a).

So there exists one-to-one correspondence between the relativistically invariant equations and the operators Γ^μ and m defined by the properties (2.5) in the space X of the representation of the Lorentz group.

3. *A Generalisation of the Generating Method*

The importance of generating invariant equations from a given one has been widely discussed by Palev (1969), who proposed a method in which, using the realisations of Lie algebras by means of Bose operators, a class of infinitely many invariant equations was found using a fixed one.

The possibility of realising any Lie algebra by means of parafield operators (Kademova, 1969; Kademova & Palev, 1970b) allows us to extend the method and to obtain infinitely many classes of equations corresponding to the choice of the parastatistics of the Green algebra.

Let us now show how, from a given invariant equation (2.4), to which the pair $(S^{\mu\nu}, \Gamma^\lambda)$ is uniquely put in correspondence, different classes of equations can be obtained.

Let the operators $S^{\mu\nu}$ and Γ^λ be given in a matrix form. Then, as is known, they can be embedded into $2n \otimes 2n$ matrices

$$\mathbf{S}^{\mu\nu} = \begin{pmatrix} S^{\mu\nu} & 0 \\ 0 & -\overset{T}{S}{}^{\mu\nu} \end{pmatrix} \qquad \mathbf{\Gamma}^\lambda = \begin{pmatrix} \Gamma^\lambda & 0 \\ 0 & -\overset{T}{\Gamma}{}^\lambda \end{pmatrix} \tag{3.1}$$

Let us remark that such matrices are contained in the algebras $sp(2n)$ and $o(n,n)$. Further, we shall denote by A_ϵ the algebra $(A_+ = sp(2n)$, $A_- = o(n,n))$ of $2n \otimes 2n$ matrices of the form:

$$L_\epsilon = \begin{pmatrix} L_\epsilon^{11} & L_\epsilon^{12} \\ L_\epsilon^{21} & L_\epsilon^{22} \end{pmatrix} \tag{3.2}$$

determined by the conditions

$$\begin{aligned} \overset{T}{L}{}_\epsilon^{22} &= -L_\epsilon^{11} \\ \overset{T}{L}{}_\epsilon^{21} &= \epsilon L_\epsilon^{21} \\ \overset{T}{L}{}_\epsilon^{12} &= \epsilon L_\epsilon^{12} \end{aligned} \tag{3.3}$$

The corresponding group $G_\epsilon(G_+ = Sp(2n), G_- = o(n,n))$ preserves a bilinear form with a matrix β

$$\beta = \begin{pmatrix} 0 & I \\ -\epsilon I & 0 \end{pmatrix}$$

Then, all the reasonings for finding some of the automorphisms of the algebra $sp(2n)$ are valid also for the algebra $o(n,n)$, and that is why we write the generated pair by $S^{\mu\nu}$ and Γ^λ in the form

$$\begin{aligned} \mathscr{S}^{\mu\nu} &= \tfrac{1}{2}\tilde{\phi}_\epsilon g_\epsilon \mathbf{S}^{\mu\nu} g_\epsilon^{-1} \phi_\epsilon \\ \mathscr{T}^\lambda &= \tfrac{1}{2}\tilde{\phi}_\epsilon g_\epsilon \mathbf{\Gamma}^\lambda g_\epsilon^{-1} \phi_\epsilon \end{aligned} \tag{3.4}$$

where

$$g_\epsilon \in G_\epsilon, \qquad \tilde{\phi}_\epsilon = (\overset{+}{A}, A), \qquad \phi_\epsilon = \begin{pmatrix} A \\ + \\ -\epsilon A \end{pmatrix}, \qquad A = (a_1, \ldots, a_n),$$

$$\overset{+}{A} = (\overset{+}{a}_1, \ldots, \overset{+}{a}_n), \qquad \epsilon \overset{+}{A} = (\epsilon \overset{+}{a}_1, \ldots, \epsilon \overset{+}{a}_n)$$

The multiplication is a matrix one. For positive ϵ the operators a_i, $\overset{+}{a}_j$ are para-Bose operators, and for negative ϵ para-Fermi ones.

In the case of $S^{\mu\nu} = S_\epsilon^{\mu\nu}$, $\Gamma^\lambda = \Gamma_\epsilon{}^\lambda \in A_\epsilon$ there is one further way of constructing $(\mathscr{S}^{\mu\nu}, \mathscr{T}^\lambda)$ by means of $n/_2$ Green operators

$$\mathscr{S}^{\mu\nu} = \tfrac{1}{2}\tilde{\phi}_\epsilon S_\epsilon^{\mu\nu} \phi_\epsilon$$
$$\mathscr{T}^\lambda = \tfrac{1}{2}\tilde{\phi}_\epsilon \Gamma_\epsilon{}^\lambda \phi_\epsilon \tag{3.5}$$

It is seen that all the formulas obtained for the generating pair $(\mathscr{S}^{\mu\nu}, \mathscr{T}^\lambda)$ by means of Bose operators are valid for much larger class of operators—para-Bose and para-Fermi operators which contain Bose operators as a particular case $\epsilon = +, p = 1$.

4. *Discussion*

The possibility of generating $\mathscr{S}^{\mu\nu}$ and $\mathscr{T}^\lambda$ by means of Green operators enables us to give realisations of the invariant equations in the space of all polynomials of parafield operators. Following Kademova & Palev (1970a) we can enlarge this space to the space of all polynomials of quasifield operators spanned on the vectors

$$(m_1{}^1, \ldots, m_n{}^p, \bar{m}_1{}^1, \ldots, \bar{m}_n{}^p) = \prod_{k=1}^{n} \prod_{\beta=1}^{p} (a_k{}^\beta)^{m_k{}^\beta} (\overset{+}{a}_k{}^\beta)^{\bar{m}_k{}^\beta} \tag{4.1}$$

and to find $(\mathscr{S}^{\mu\nu}, \mathscr{T}^\lambda)$ in a matrix form. For some equations it is convenient to use the space of all polynomials of only creation quasifield operators.

It is of interest to study in detail the different equations obtained in this way, especially those which can be obtained by the use of the Fermi operators $\epsilon = -, p = 1$. However, in this paper we restrict ourselves to these general remarks without studying any example.

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